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THE INFLUENCE OF THE APERTURE STOP FRACTAL SHAPE OF AN OPTICAL SYSTEM ON THE ILLUMINANCE DISTRIBUTION

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Abstract

In the article the options for the application of aperture shapes with fractal properties in the design of optical systems are considered. Calculations of mathematical models of point spread functions of a diffraction-limited optical system are performed. The diffraction patterns of the light distribution in these systems are presented, and the point spread functions are considered for various shapes of the aperture stop. Analytical expressions are obtained for the light distribution depending on the pupil shape, which can be used to control the process of image formation. The pupil shape, which has the shape of an equilateral triangle, is chosen as the basic one, and the shape of the pupil as a "Koch snowflake" curve is also considered. Using the Fraunhofer integral, the dependences of the distribution of the spectral density of the complex amplitude on the aperture located on an opaque screen are derived in the Fraunhofer approximation and under the condition of illumination by a plane monochromatic wave. Using the relationship with the complex amplitude, the sought-for intensity distribution in the plane of the diffraction pattern is obtained. Taking into account the simplifications adopted in this article, the solution of the Fraunhofer integral is found, by setting the integration limits, depending on: the selected aperture profile, the coordinate system chosen for it, and the position of nodal points in this system

Keywords

Aperture stop, aperture, point spread function, Fraunhofer diffraction, fractal

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Introduction. There is a range of practical tasks that require optical systems with the particular intensity distribution of point source image, when it is necessary to provide a "smooth" distribution of energy rather than minimum spot size. When it comes to practice, optical systems with circular apertures are used in most cases. Nevertheless, there are optical systems with apertures of more complex shapes [1], which require evaluating their impact on image quality.

Formulation of the problem. In this case, a possible option could lie in using an aperture of a particular shape, on account of which an appropriate point spread function (PSF), which characterizes system resolution [2–12], will be formed in an optical system. Particularly, in our opinion, a fractal structure may be a suitable shape, as it is demonstrated in studies [13–17]. The possibility of using suchlike shapes when designing an optical system allows to laying properties that affect system resolution in advance before the synthesis of its components. This way the necessity of complicating the optical system is eradicated. However, the main difficulty of taking the approach lies in getting the analytical dependencies needed for the calculation of light distribution passing through such an aperture. The process of forming a picture could be regulated with their help.

This study aims is to examine the impact of the aperture's shape on the formation of the point spread function of a diffraction-limited optical system.

Applicability condition, assumptions, problem-solving method. The solution to the problem is viewed in the initial approximation. We regard the original optical system as a diffraction-limited one which forms an aberrationless image of a point source in the Fraunhofer diffraction approximation. A flat wavefront from a distant monochromatic source is incident on the aperture stop. The aperture stop plane is considered an opaque screen with a hole, and the plane of image formation is taken as the observation plane or the plane of the diffraction pattern. Spectral density distribution of complex amplitude versus aperture on an opaque screen during Fraunhofer diffraction and illumination by a plane monochromatic wave (Fig. 1), according to [18, 19], is defined by the formula:

$$\tilde{U}\left(\nu_x, \nu_y\right) \sim \int\int_{-\infty}^{\infty} U(x, y) \exp\left[-j 2\pi\left(\nu_x x + \nu_y y\right)\right] dx\, dy, \qquad (1)$$

where $\nu_x = \psi_x/\lambda = \xi/\left(\lambda z_0\right)$, $\nu_y = \psi_y/\lambda = \eta/\left(\lambda z_0\right)$ are spatial frequencies; $U(x, y)$ is complex wave amplitude at a point $P(x, y)$; λ is wavelength; ψ_x, ψ_y are diffraction angles; z_0 is distance between planes.

In the calculations, we make the following assumptions: the size of the hole on the screen and the size of the observation area are much smaller than the distance from the screen to the observation plane; the size of the hole on the screen is large compared to the wavelength of the incident light, and the light source is monochromatic and located at such a distance from the screen that the light incident on the screen has an almost flat wavefront and a constant amplitude. Therefore, for a monochromatic plane wave we take $U(x, y) = E = \text{const}$.

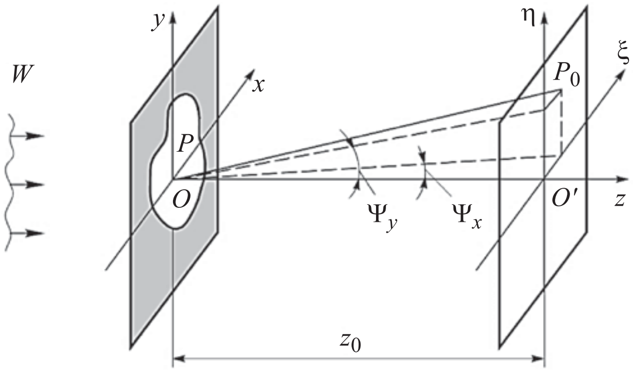

Fig. 1. Diffraction of a monochromatic wave in the general case

The advantage of the Fraunhofer diffraction approximation lies in the correlation of spatial frequencies with diffraction angles, which allows one to make the transition from spectral density to the most complex amplitude by the corresponding change of coordinates, rather than calculating the inverse Fourier transform instead. Thus, speaking of the distribution of the spectral density of the complex amplitude from the aperture, we can also judge the distribution of the complex field amplitude in the plane of the diffraction pattern. Similar reasoning is used for the intensity distribution.

Accordingly, the intensity distribution in the diffraction pattern plane is associated with the distribution of the complex amplitude by the known expression [18, 19]:

$$\tilde{I}\left(\nu_x,\, \nu_y\right) = \tilde{U}\left(\nu_x,\, \nu_y\right)\tilde{U}^*\left(\nu_x,\, \nu_y\right), \qquad (2)$$

where $\tilde{U}\left(\nu_x,\, \nu_y\right)$ is complex conjugate spectrum of complex amplitude.

Thus, the solution to the diffraction problem is reduced to finding the integral (1) and substituting the resulting expression in the formula (2). Then, the corresponding coordinate replacement is carried out considering the diffraction approximation, and the intensity distribution function in the image of a point source, i.e., point spread function of a diffraction-limited system, is obtained. The solution of integral (1), taking into account the correctly specified integration limits, is determined by the aperture profile and the reference coordinate system selected for it.

Calculation of Fraunhofer diffraction by triangle aperture. At first, we consider the shape of an equilateral triangle as the basic shape of the aperture. Within the normal incidence of a plane wave on an equilateral triangular hole (Fig. 2), we have the coordinates of points *1* and *2*.

The Euler formula has to be used to calculate the integral (1). We choose a counterclockwise direction of travel (positively oriented contour), and then apply Green's theorem [20]. The closed-loop integral is found as the sum of three integrals taken on each side of the triangle. After calculating each of the three amplitudes and their addition, we obtain the intensity distribution according to the formula (2):

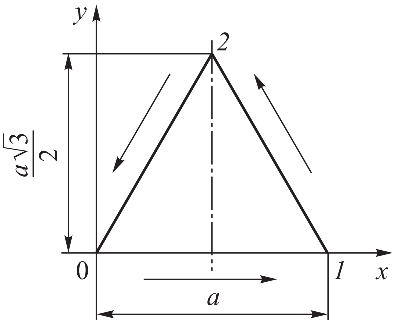

Fig. 2. Triangle aperture

$$\tilde{I}(\nu_x,\nu_y)=\frac{E^2a^2}{16\pi^2}\left\{\frac{3}{\nu_x^2}\left[\sin c\left(\frac{a\pi}{2}\left(\nu_x-\sqrt{3}\nu_y\right)\right)\times \sin\left(\frac{\sqrt{3}a\pi}{2}\left(\nu_y+\sqrt{3}\nu_x\right)\right)-\sin\left(\frac{a\pi}{2}\left(\nu_y+\sqrt{3}\nu_x\right)\right)\times \sin c\left(\frac{a\pi}{2}\left(\nu_x+\sqrt{3}\nu_y\right)\right)\right]^2+\frac{1}{\nu_y^2}\left[\cos\left(\frac{\sqrt{3}a\pi}{2}\left(\nu_y+\sqrt{3}\nu_x\right)\right)\times \sin c\left(\frac{a\pi}{2}\left(\nu_x-\sqrt{3}\nu_y\right)\right)+\sin c\left(a\pi\left(\nu_x+\sqrt{3}\nu_y\right)\right)-2\sin c\left(a2\pi\nu_x\right)\right]^2\right\}. \tag{3}$$

In formula (3), we take the natural logarithm of the expression for the so-called "amplitude amplification effect". In addition, if a graph of level lines (top view in the sagittal plane) of the obtained dependence is made, then it is called the diffraction pattern of the light distribution in the image plane.

The graph of the tangential section was built using the expression (3) with the replacement of spatial frequencies by spatial coordinates denoted as $x'(\xi)$ and $y'(\eta)$ respectively, where $y' = 0$, is called the point spread function (PSF). The same graph characterizes the parameters of the scattering spot. All dependencies are normalized in amplitude.

Calculation of the Fraunhofer diffraction from the fractal aperture — "Koch snowflake". The mechanism of formation of the fractal is shown in Fig. 3 and presented in [21].

We assume that the initial triangle with sides a (see Fig. 2), the distribution of the field from which is already known, is the zero iteration count.

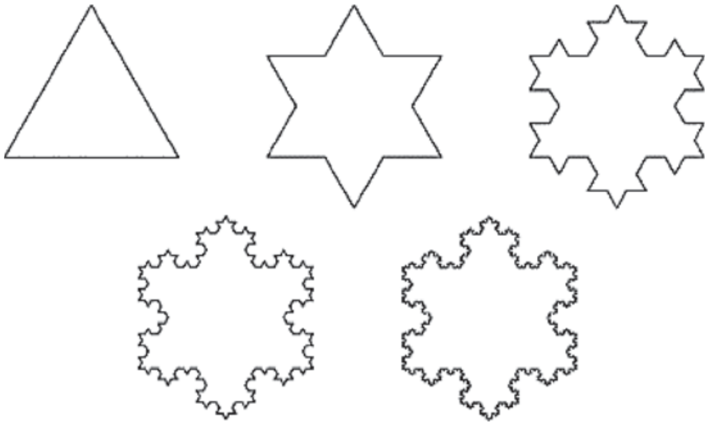

Fig. 3. The “Koch snowflake” formation (4 iteration steps)

Therefore, to find the intensity distribution formula in the case of the first iteration, it is necessary to consider the formation of a six-pointed star from the initial triangle (Fig. 4) and find the field distribution from it, that is, the field distribution from three more triangles in addition to the original one.

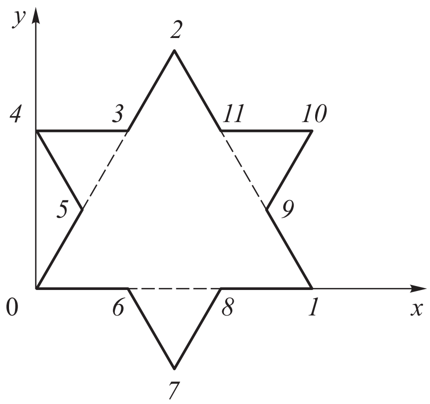

Fig. 4. Formation of a six-pointed star from the initial triangle

The distributions of light intensities in the image plane from diffraction by triangles *3-4-5*, *6-7-8*, and *9-10-11*, calculated by the formula (1), are identical to each other.

In the subsequent formation of new triangles (the following iterations), the self-similarity of the field distribution from the fractal can be verified. Therefore, we formulate the final light distribution intensity during diffraction from the hole in the shape of the “Koch snowflake” curve of the *n*-th iteration, considering the number of newly formed triangles at each step *n*:

$$\tilde{I}_{\Sigma}\left(\nu_x, \nu_y\right) = \frac{3a^2E^2}{16\pi^2\nu_x^2}\left\{\sin c^2\left[\frac{\pi a}{2}\left(\nu_x + \sqrt{3}\nu_y\right)\right] + \right.$$

$$+ \sin c^2\left[\frac{\pi a}{2}\left(\nu_x - \sqrt{3}\nu_y\right)\right] - 2\sin c\left[\frac{\pi a}{2}\left(\nu_x + \sqrt{3}\nu_y\right)\right] \times$$

$$\left. \times \sin c\left[\frac{\pi a}{2}\left(\nu_x - \sqrt{3}\nu_y\right)\right]\cos\left(\pi a \nu_x\right)\right\} +$$

$$+\sum_{n=1}^{\infty}\left[\frac{2^{2(n-3)}a^2E^2}{9^{n-1}\pi^2\nu_x^2}\left\{\sin c^2\left[\frac{\pi a}{2\cdot 3^n}\left(\nu_x+\sqrt{3}\nu_y\right)\right]+\right.\right.$$

$$+\sin c^2\left[\frac{\pi a}{2\cdot 3^n}\left(\nu_x-\sqrt{3}\nu_y\right)\right]-2\sin c\left[\frac{\pi a}{2\cdot 3^n}\left(\nu_x+\sqrt{3}\nu_y\right)\right]\times$$

$$\left.\left.\times\sin c\left[\frac{\pi a}{2\cdot 3^n}\left(\nu_x-\sqrt{3}\nu_y\right)\right]\cos\left(\frac{\pi a}{2\cdot 3^n}\nu_x\right)\right\}\right]. \quad (4)$$

The diffraction pattern is shown in Fig. 5. As you can see, the distribution structure is uniform in all directions. The given distributions in all figures are calculated at a wavelength of $\lambda = 700$ nm, side $a = 2$ μm and amplitude $E = 10^6$.

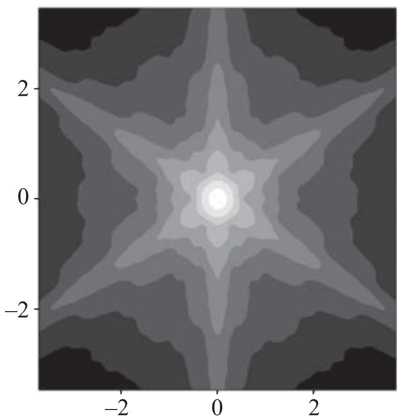

Fig. 5. Diffraction pattern of light distribution from the aperture in the shape of a “Koch snowflake” of the 4th iteration

Discussion of the obtained results. A comparison of the point spread functions for each of the cases considered is carried out with a classic round aperture without shielding and a rectangular aperture. We take the radius of the circumference of the round hole equal to $R = a/\sqrt{3}$ and assume that both the triangular shape of the pupil and the fractal shape of the “Koch snowflake” curve are inscribed in the same circle. The intensity distribution from the round pupil in the Fraunhofer diffraction is written as [18]:

$$\tilde{I}(\nu_x, \nu_y) = \left(E\pi R^2\frac{2J_1(2\pi\nu_r R)}{2\pi\nu_r R}\right)^2, \quad (5)$$

where $\nu_r = \sqrt{\nu_x^2 + \nu_y^2}$ is radial spatial frequency.

Accordingly, the intensity distribution from the rectangular pupil in the Fraunhofer diffraction is

$$\tilde{I}(\nu_x, \nu_y) = \left(Eab\right)^2 \sin c^2\left(a\pi\nu_x\right)\sin c^2\left(b\pi\nu_y\right), \quad (6)$$

where the ratio of length a to width b in the rectangle is 5 : 4.

The point spread functions of the diffraction-limited system for each shape of the aperture are presented, according to the definition given above, in Fig. 6. As can be seen in Fig. 6, the first PSF minimum for a triangular hole coincides

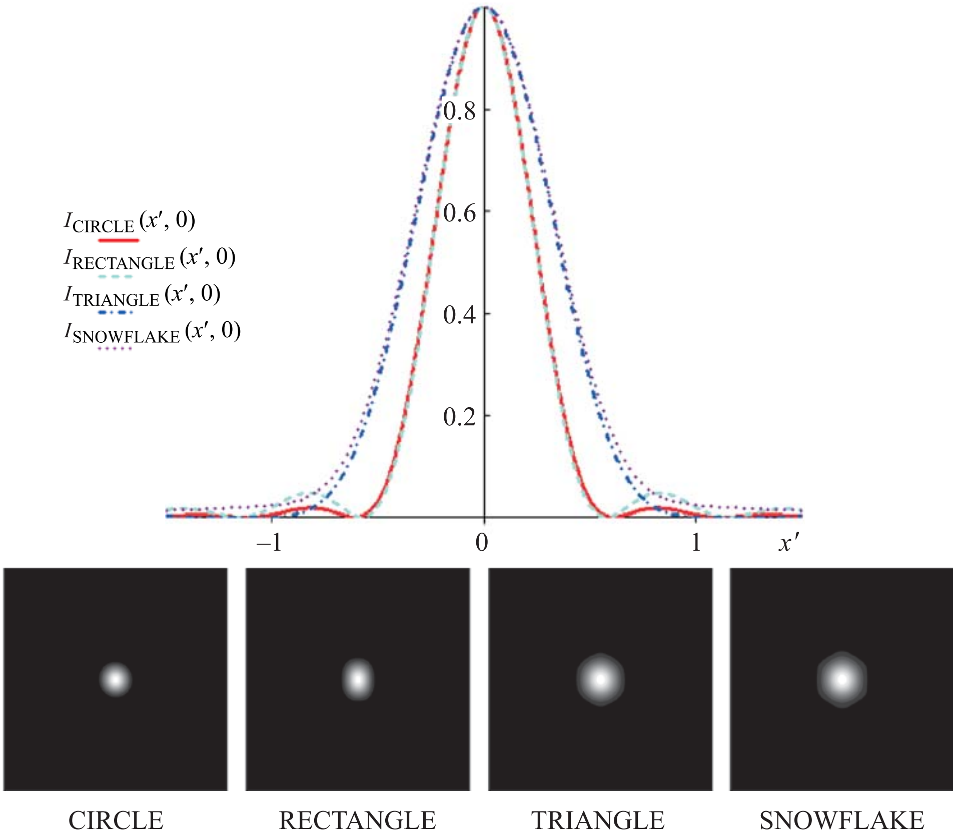

Fig. 6. Point spread functions for cases when the aperture is round, rectangular, triangular and in the shape of the "Koch snowflake"

with the second PSF minimum for the round one, the scattering spot from the triangular hole is larger, its distribution is more uniform, and it reduces the effect of secondary maxima relative to the PSF for a circular aperture, which can be used in application packages as an image noise filtering function [2, 22, 23].

In the case of the "Koch snowflake", we have an even larger scattering spot than the PSF for the triangle, while the PSF for the snowflake goes around the maxima of the PSF for the circle and is even more uniform than for the triangle. PSF from a rectangular hole has specific large secondary maxima, which are larger than in the rest of the cases considered.

Thus, the shape of the aperture entails a specific change in the scattering spot, which allows one to use PSF to analyze the properties and image quality of the optical system [11, 12, 24, 25]. When giving the aperture shape fractal properties ("fractalization" of the shape) we have a smoother intensity distribution in the image space. Such intensity distribution can be applied, for example, in portrait lenses or laser systems, where high-resolution structures can play a negative role for the user and, therefore, it may be necessary to "blur" the image (increase the scattering spot). To do so, it is unnecessary to bring in defocusing of the system, change its design and dimensions, or achieve an exponential decrease in the amplitude transmittance function, which also leads to a more uniform intensity distribution and an increase in the scattering spot. It is enough to replace the shape of the aperture stop from round to fractal, which speeds up the system design process to some extent.

Conclusion. A mathematical description of the point spread function formation process in the Fraunhofer diffraction approximation is proposed. A comparative analysis of the intensity distribution in the image of a point for a round, triangular and fractal ("Koch snowflake") apertures is carried out and it is proved that the fractal aperture shape can be used to increase the uniformity of the image illumination in the optical system.

Based on the developed methodology, it is possible to carry out a preliminary analysis of image quality by the described point spread functions even before designing the optical system itself.

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